Set Theory Revisited

Outline for Today

- ***Set Theory Definitions***
  - Making our intuitions rigorous.
- ***Proofs on Sets***
  - How to reason about groups of items.
- ***Images and Preimages***
  - Generalizing functions to sets.

Recap from Last Time

	To ***prove*** that this is true...	If you ***assume*** this is true...
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	Introduce a variable x into your proof that has property A.
$A \to B$	Assume A is true, then prove B is true.	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.
$A \land B$	Prove A. Then prove B.	Assume A. Then assume B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*	Consider two cases. Case 1: A is true. Case 2: B is true.
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	Assume $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

New Stuff!

Thinking About Sets

Reasoning About Sets

- When working with sets and set operators, your intuition might be to think about the sets themselves.
  - e.g. if A is the set of all animals and S is the set of all smiling things, then $A \cap S$ is the set of all smiling animals.
- However, when writing mathematical proofs, this is usually ***not*** the approach we take.
- Instead, we focus on the individual elements that make up those sets and properties of those individual elements.

If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generations of creatures, what statement would contain the most information in the fewest words? I believe it is the atomic hypothesis [...] that ***all things are made of atoms*** [. ...] In that one sentence, you will see, there is an enormous amount of information about the world, if just a little imagination and thinking are applied.

-Richard Feynman

All sets are made of elements, and they're completely defined by their elements.

	Is defined as…	To ***prove*** that this is true…	If you ***assume*** this is true…
$S \subseteq T$	$\forall x \in S.\ x \in T$	Pick an arbitrary $x \in S$. Prove $x \in T$	Initially, ***do nothing***. Once you find some $x \in S$, conclude $x \in T$.
$S = T$	$S \subseteq T \land T \subseteq S$	Prove $S \subseteq T$. Then prove $T \subseteq S$.	Assume $S \subseteq T$ and $T \subseteq S$.
$x \in A \cap B$	$x \in A \land x \in B$	Prove $x \in A$. Then prove $x \in B$.	Assume $x \in A$. Then assume $x \in B$.
$x \in A \cup B$	$x \in A \lor x \in B$	Either prove $x \in A$ or prove $x \in B$.	Consider two cases: Case 1: $x \in A$. Case 2: $x \in B$.
$X \in \wp(A)$	$X \subseteq A$.	Prove $X \subseteq A$.	Assume $X \subseteq A$.
$x \in \{ y \mid P(y) \}$	$P(x)$	Prove $P(x)$.	Assume $P(x)$.

Putting This Into Practice

Theorem: Let A, B, C, and D be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

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Proof: Pick some $x \in A \cup B$; we need to show that $x \in C \cup D$.

Because $x \in A \cup B$, we know that $x \in A$ or $x \in B$. We consider each case separately.

Case 1: $x \in A$. Since $A \subseteq C$ and $x \in A$, we see that $x \in C$, and therefore that $x \in C \cup D$.

Case 2: $x \in B$. Then because $B \subseteq D$ and $x \in B$ we have $x \in D$, so $x \in C \cup D$.

In either case, we see that $x \in C \cup D$, as required. ■

Theorem: Let A and B be sets. Then if $\wp(A) = \wp(B)$, then $A = B$.

Theorem: Let A and B be sets. If $\wp(A) = \wp(B)$, then $A = B$.

Proof: Assume $\wp(A) = \wp(B)$. We need to show that $A = B$, or, equivalently, that $A \subseteq B$ and $B \subseteq A$. Since the roles of A and B are symmetric, we'll just prove $A \subseteq B$.

Pick some $x \in A$; we need to show that $x \in B$. Since $x \in A$, we know that $\{x\} \subseteq A$. This means that $\{x\} \in \wp(A)$, and since $\wp(A) \subseteq \wp(B)$ we know $\{x\} \in \wp(B)$. Thus we see that $\{x\} \subseteq B$, which in turn means that $x \in B$, as required. ■

There are several other ways to prove this result. See the appendix for some cute alternative proofs!

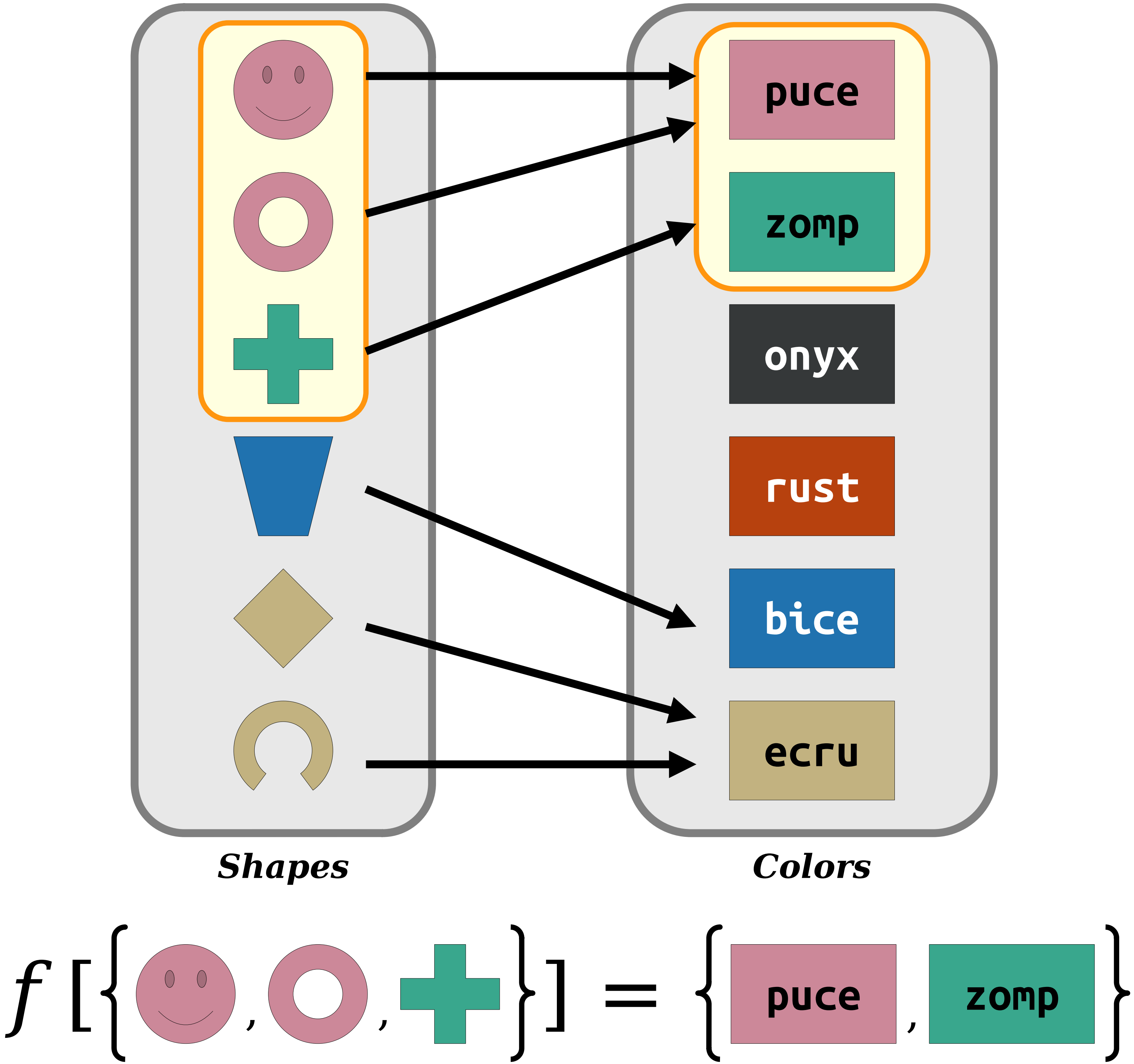
Time-Out for Announcements!

Problem Set Three

- PS2 was due today at 1:00PM.
  - Need more time? Use a late day to extend the deadline to 1:00PM tomorrow.
- PS3 goes out today. It’s due next Friday at 1:00PM.
  - Explore properties of functions and the nature of infinity!
  - Prove results that have applications deep within CS theory and practice!
- As usual, feel free to stop by office hours or post on EdStem if you have any questions!

Back to CS103!

Images and Preimages



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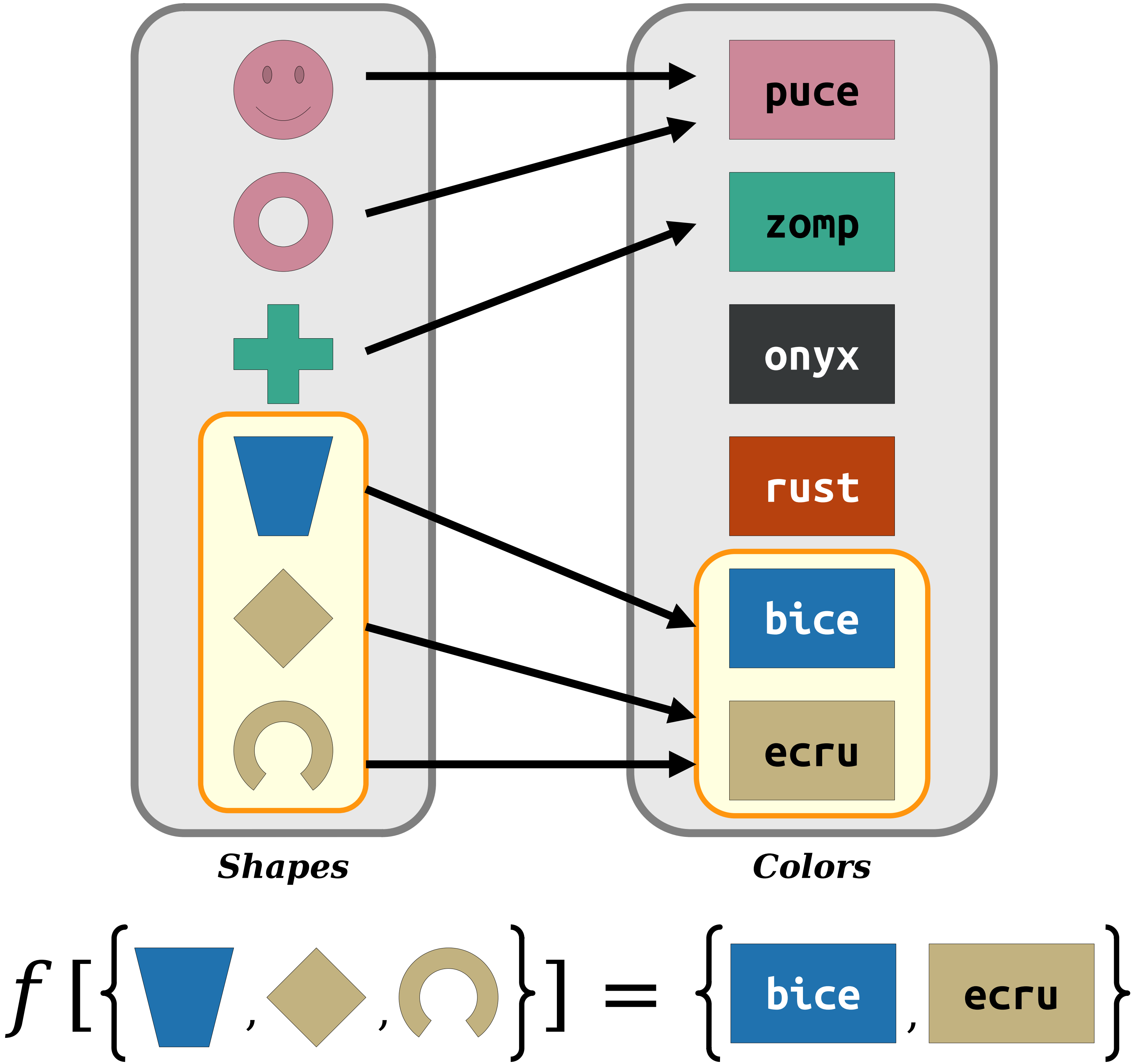
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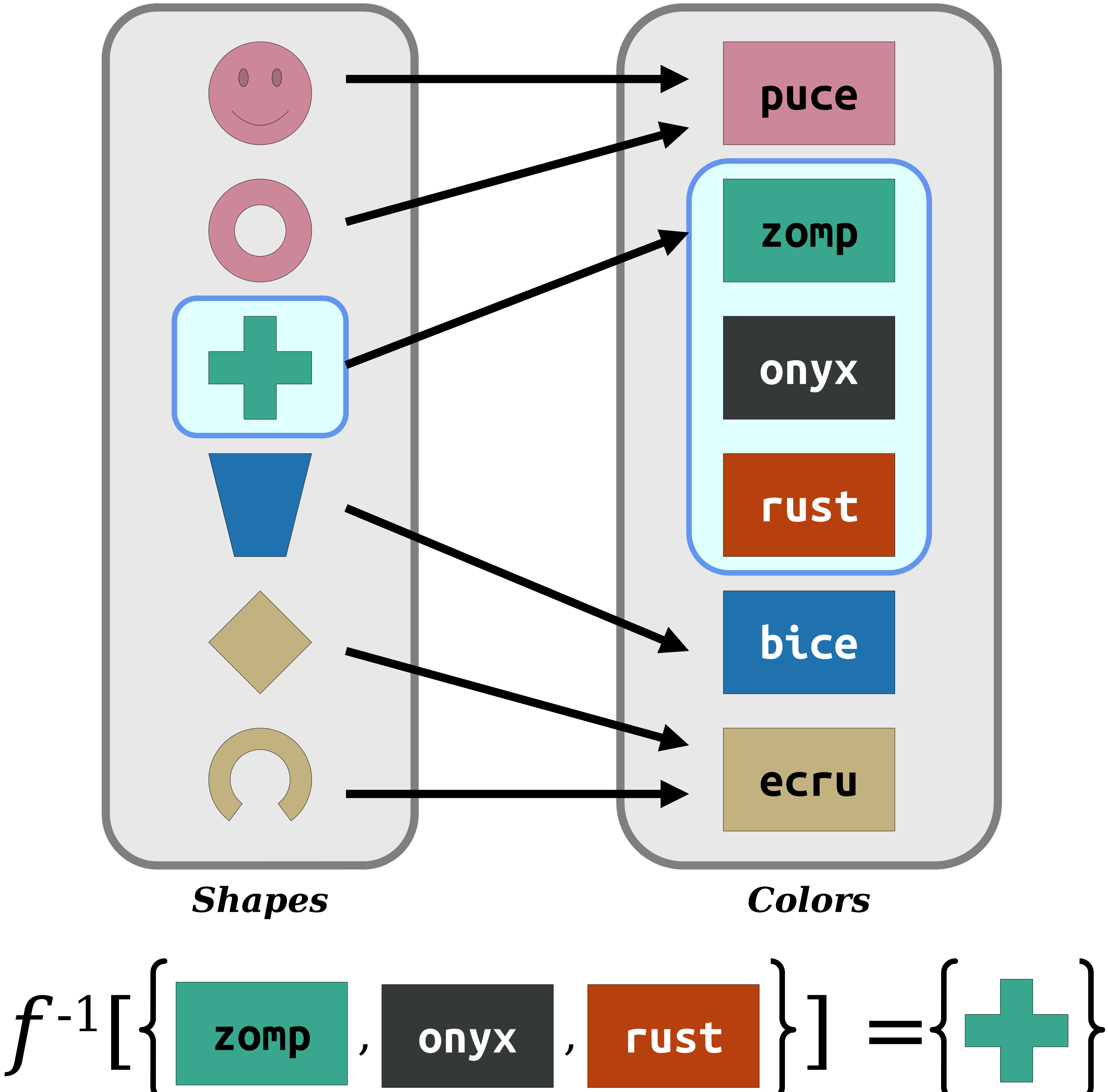

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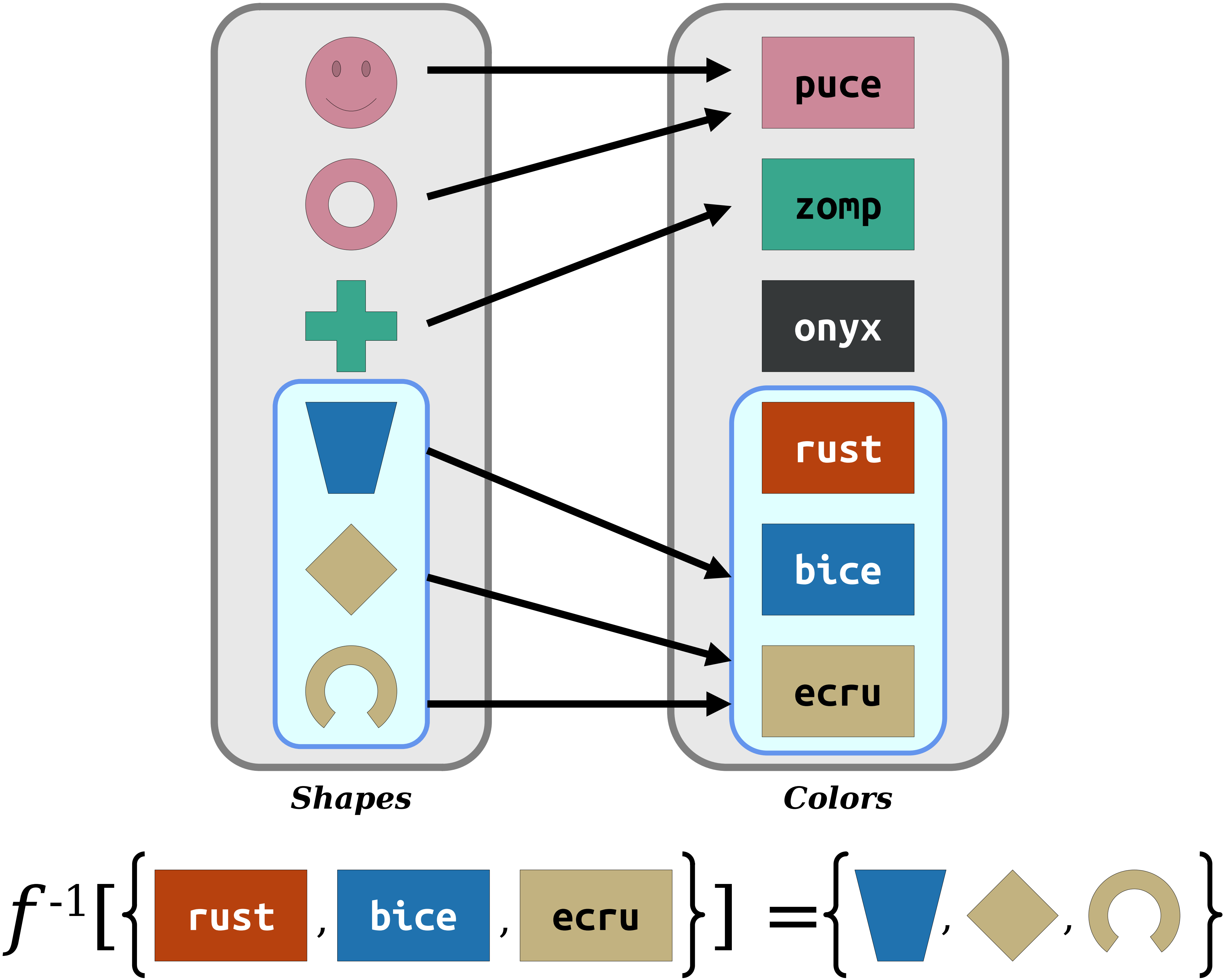
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Images and Preimages

- Let $f : A \to B$ be a function.
- For any set $X \subseteq A$, the ***image of X under f***, denoted $\boldsymbol{f[X]}$, is the set

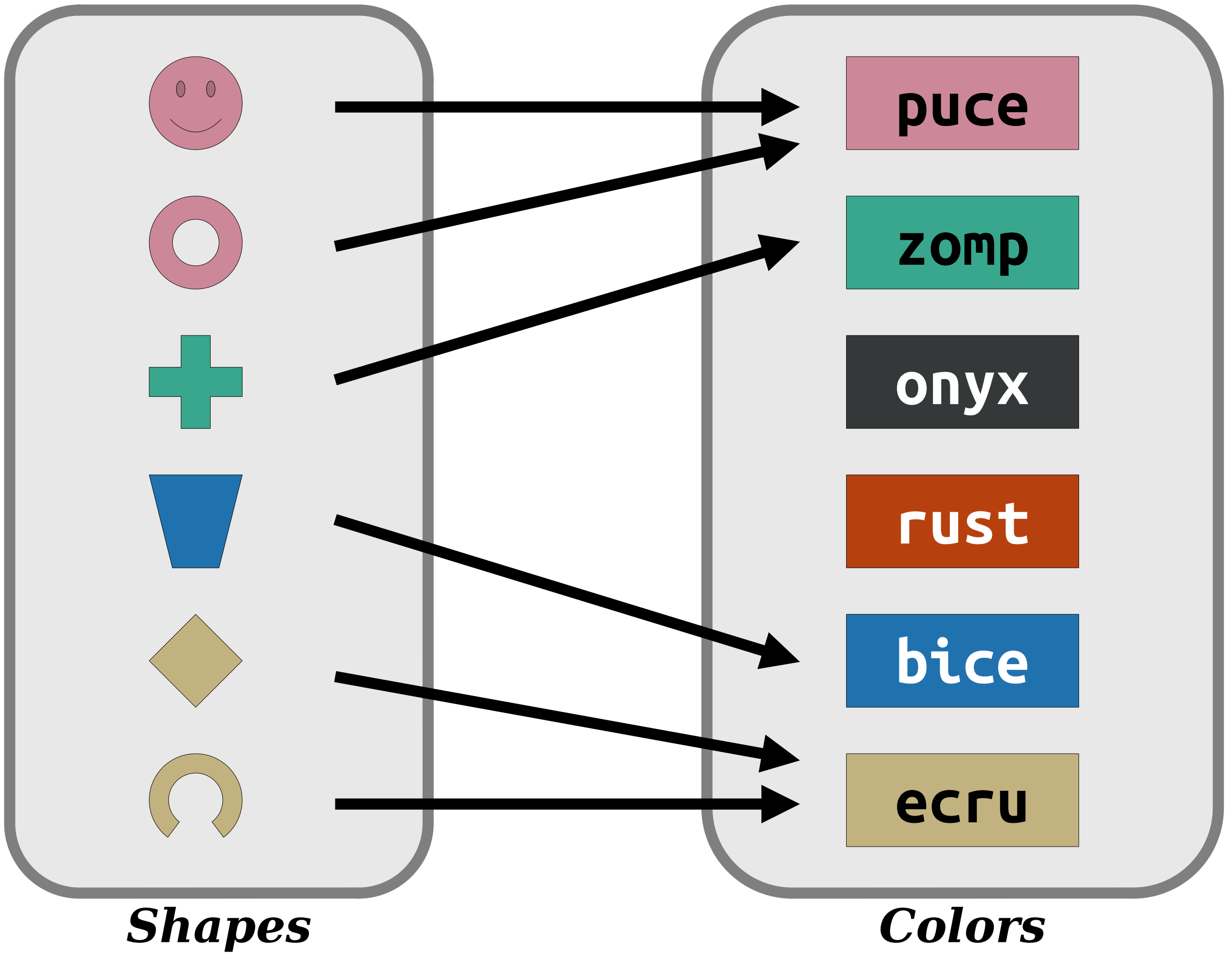
$$f[X] = \{\ b \mid \exists a \in X.\ f(a) = b\ \}.$$

(*"The set of all codomain elements mapped to by the set X."*)

- For any set $Y \subseteq B$, the ***preimage of Y under f***, denoted $\boldsymbol{f^{-1}[Y]}$, is the set

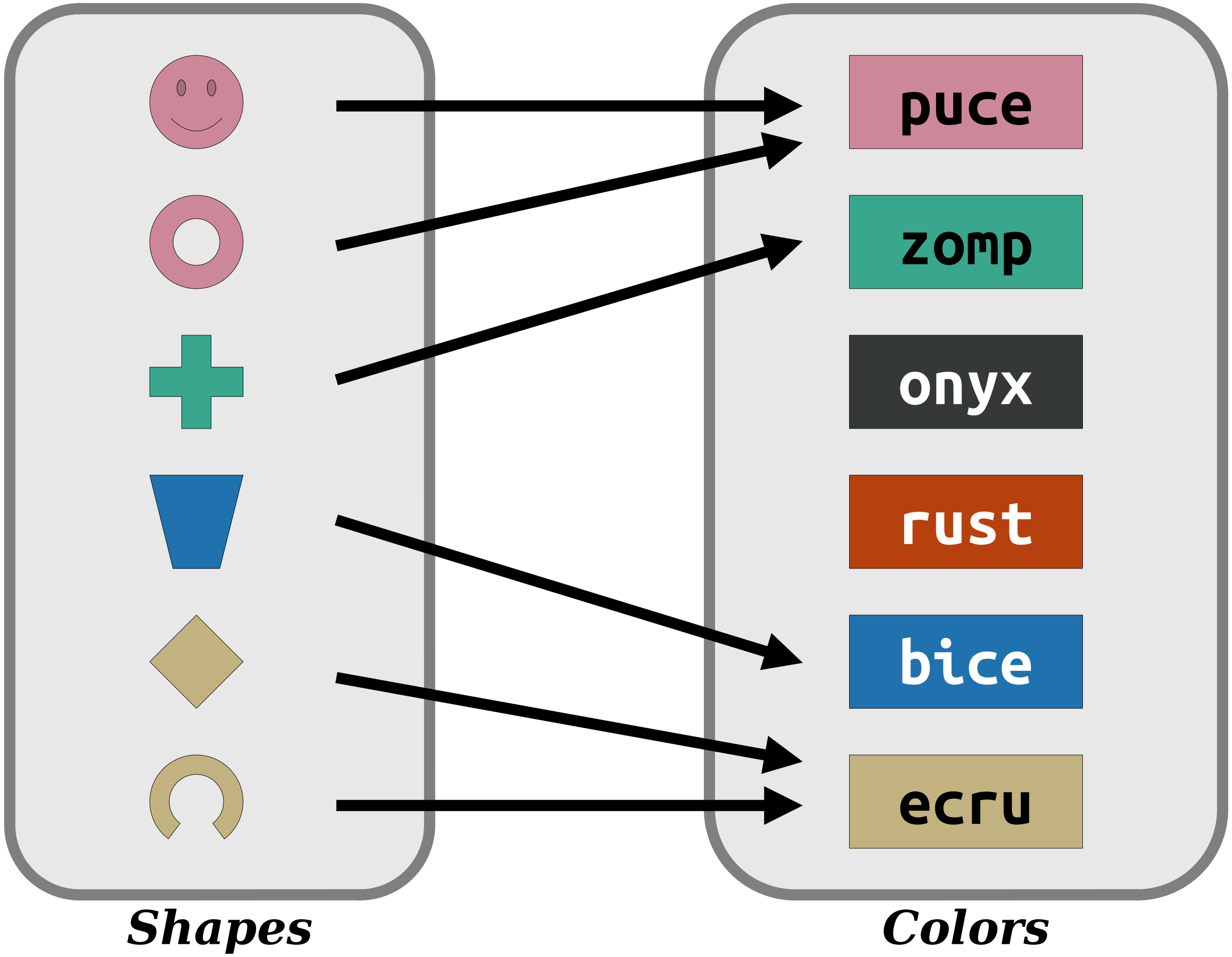
$$f^{-1}[Y] = \{\ a \mid a \in A \land f(a) \in Y\ \}$$

(*"The set of all domain elements that map to something in Y."*)



What is $f[Shapes]$?
What is $f^{-1}[Colors]$?

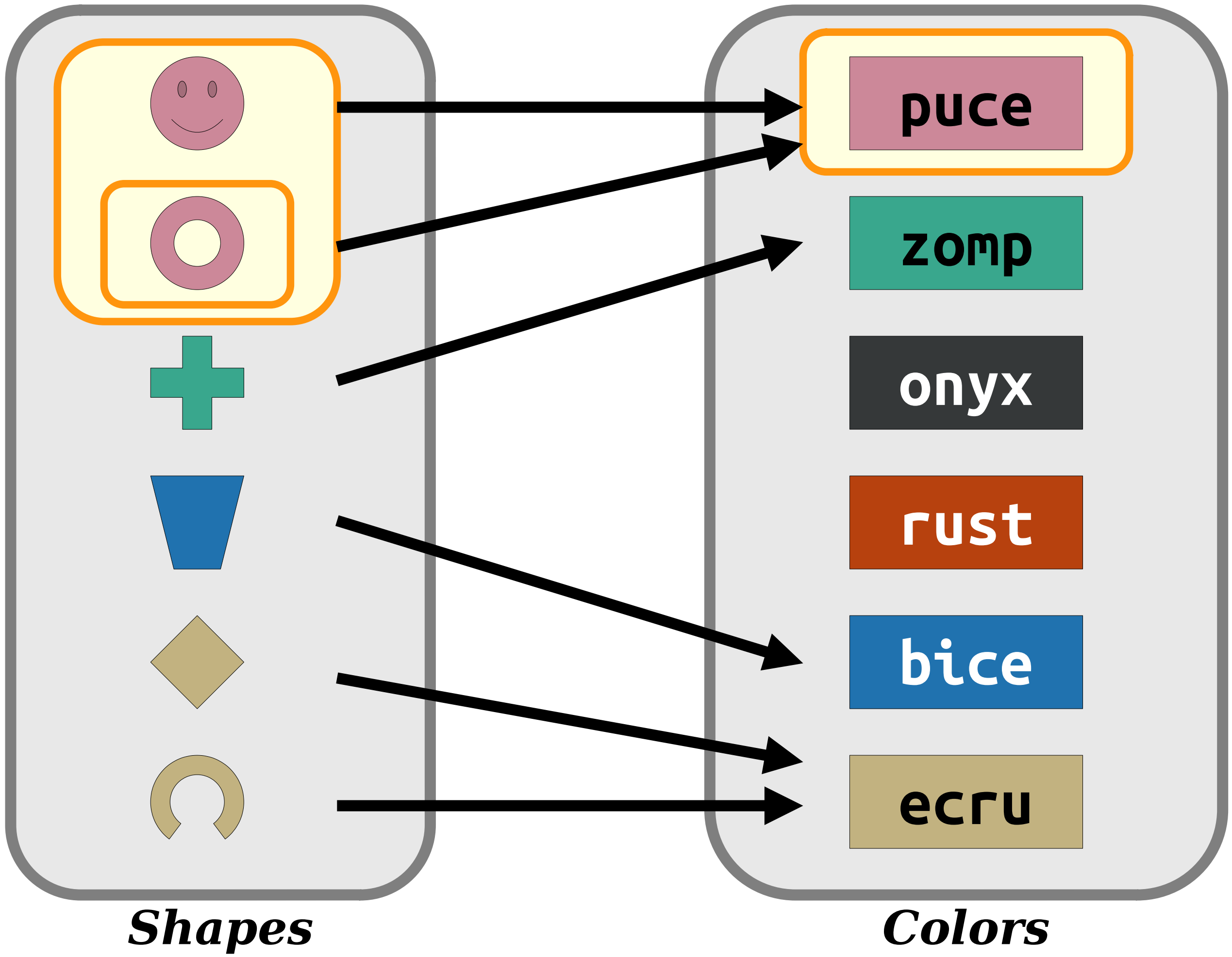
Answer at
https://pollev.com/cs103aut23



Are the following true?

$$\forall X \in \wp(Shapes).\ f^{-1}[f[X]] = X$$

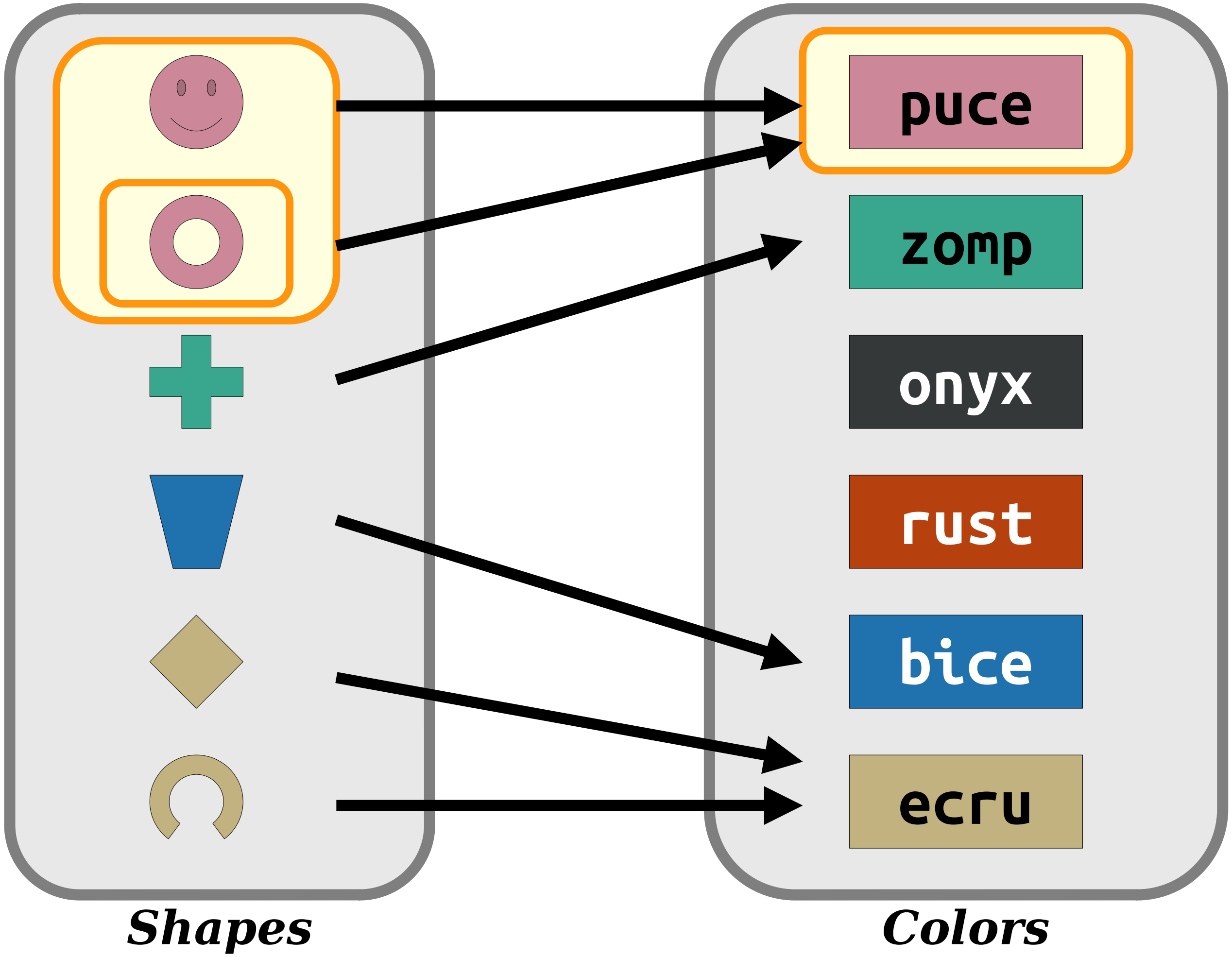
$$\forall Y \in \wp(Colors).\ f[f^{-1}[Y]] = Y$$



Are the following true?

~~$\forall X \in \wp(\mathit{Shapes}).\ f^{-1}[f[X]] = X$~~

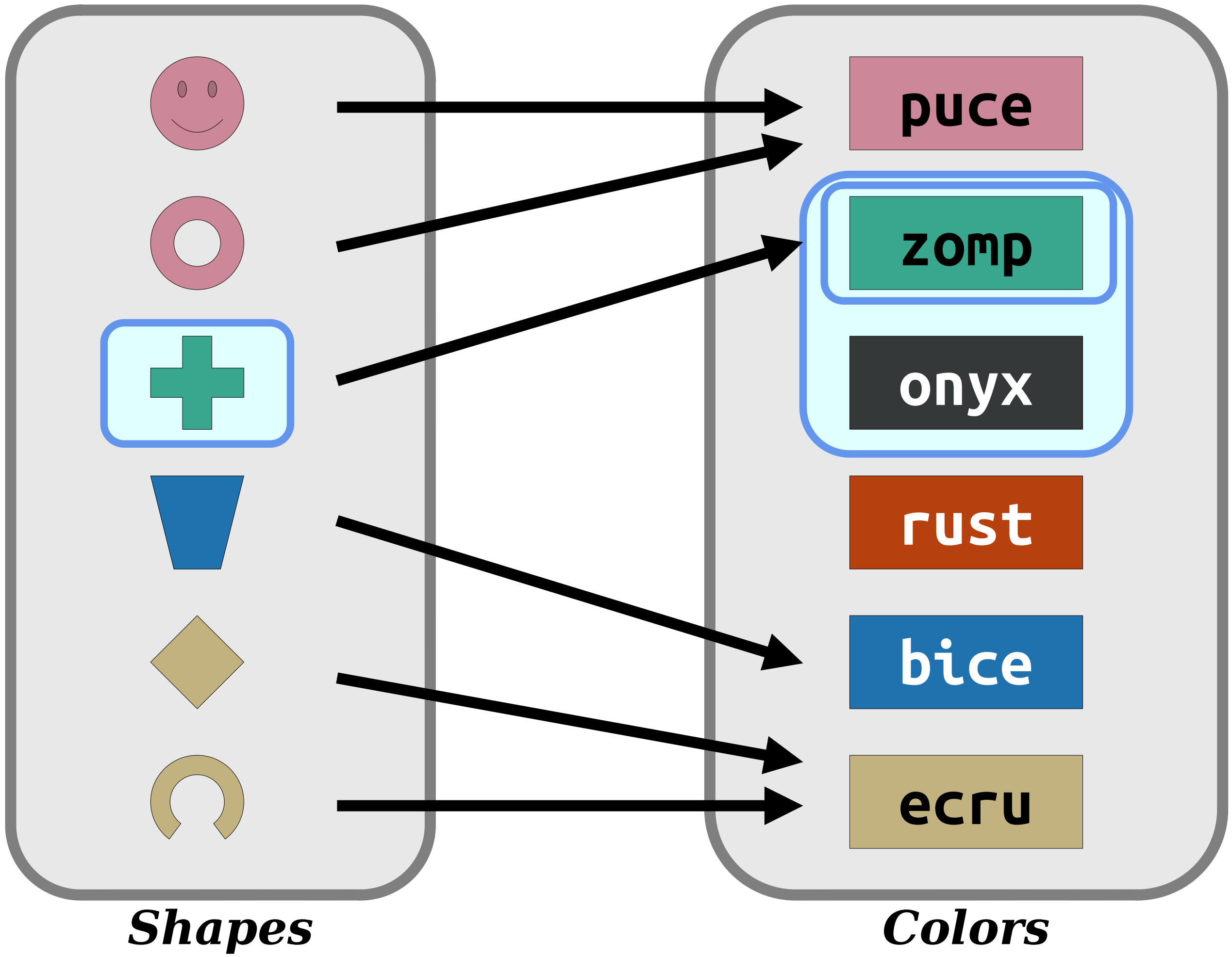
$\forall Y \in \wp(\mathit{Colors}).\ f[f^{-1}[Y]] = Y$



Are the following true?

$\forall X \in \wp(Shapes).\ X \subseteq f^{-1}[f[X]]$

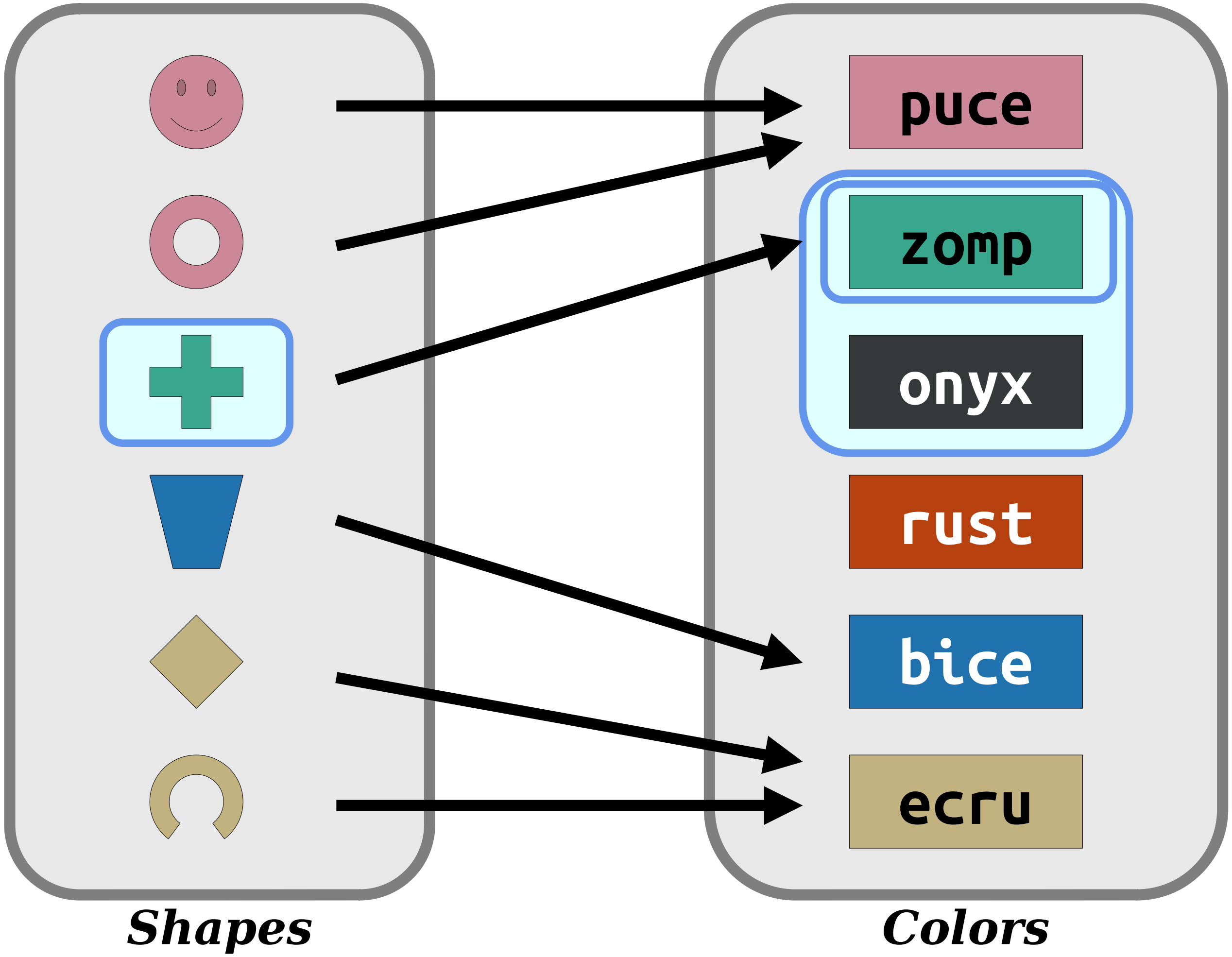
$\forall Y \in \wp(Colors).\ f[f^{-1}[Y]] = Y$



Are the following true?

$$\forall X \in \wp(Shapes).\ X \subseteq f^{-1}[f[X]]$$

~~$$\forall Y \in \wp(Colors).\ f[f^{-1}[Y]] = Y$$~~



Are the following true?

$$\forall X \in \wp(Shapes).\ X \subseteq f^{-1}[f[X]]$$

$$\forall Y \in \wp(Colors).\ \ f[f^{-1}[Y]] \subseteq Y$$

Theorem: Let $f : A \to B$ be a function. Then for any $Y \subseteq B$, we have $f[f^{-1}[Y]] \subseteq Y$.

Theorem: Let $f : A \to B$ be a function. Then for any $Y \subseteq B$, we have $f[f^{-1}[Y]] \subseteq Y$.

Proof: Choose some $Y \subseteq B$. We need to show that $f[f^{-1}[Y]] \subseteq Y$. To do so, choose $b \in f[f^{-1}[Y]]$ arbitrarily, and we'll show that $b \in Y$.

Since $b \in f[f^{-1}[Y]]$, we know that there exists an $a \in f^{-1}[Y]$ where $f(a) = b$. And because $a \in f^{-1}[Y]$, we learn that $f(a) \in Y$. Finally, since $b = f(a)$ and $f(a) \in Y$, we have that $b \in Y$, as required. ■

Theorem: Let $f : A \to B$ be a function. Then for any $X \subseteq A$, we have $X \subseteq f^{-1}[f[X]]$.

Proof: Appendix!

Theorem: Let $f : A \to B$ be a injection. Then for any $X \subseteq A$, we have $X = f^{-1}[f[X]]$.

Theorem: Let $f : A \to B$ be a surjection. Then for any $Y \subseteq B$ we have $Y = f[f^{-1}[Y]]$.

Proof: See appendix!

To Summarize

- Proofs on sets are almost always, though not universally, proofs about elements of those sets.
- Operations on sets are almost always defined in terms of properties of their elements. Those definitions guide proofs about sets and set theory.
- Images and preimages let us apply functions to groups of elements, or think about what groups of elements would give us some result.

Next Time

- ***Graph Theory***
  - A ubiquitous, powerful abstraction with applications throughout computer science.
- ***Vertex Covers***
  - Making sure tourists don't get lost.
- ***Independent Sets***
  - Helping the recovery of the California Condor.

Appendix: More proofs on sets.

Theorem: Let $f : A \to B$ be a function. Then for any $X \subseteq A$, we have $X \subseteq f^{-1}[f[X]]$.

Proof: Choose some $X \subseteq A$. We need to show that $X \subseteq f^{-1}[f[X]]$. So choose some $a \in X$; we must prove that $a \in f^{-1}[f[X]]$.

To show that $a \in f^{-1}[f[X]]$, we need to show that $a \in A$ and that $f(a) \in f[X]$. Because $a \in X$ and $X \subseteq A$, we already know that $a \in A$. Thus we just need to prove that $f(a) \in f[X]$. To that end, we just need to show there is an $x \in X$ where $f(a) = f(x)$. And that's not too bad; we just pick $x = a$ and call it a day. ■

Theorem: Let A and B be sets. Then if $\wp(A) = \wp(B)$, then $A = B$.

In lecture, we proved this using a direct proof. Here are two other ways to prove this result, one also done directly and one done via contrapositive.

Theorem: Let A and B be sets. Then if $\wp(A) = \wp(B)$, then $A = B$.

Proof 2: Assume $\wp(A) = \wp(B)$. We need to show that $A = B$. Equivalently, we will show that $A \subseteq B$ and $B \subseteq A$.

First, we note that $A \subseteq A$. (To see this, pick any $x \in A$; then $x \in A$ as well.) This means that $A \in \wp(A)$. Then, since $\wp(A) = \wp(B)$, we see that $A \in \wp(B)$, so $A \subseteq B$.

The proof that $B \subseteq A$ is the same as above, but with the roles of A and B reversed. ■

Theorem: Let A and B be sets. Then if $\wp(A) = \wp(B)$, then $A = B$.

Proof 3: We will prove the contrapositive; namely, that if $A \neq B$, then $\wp(A) \neq \wp(B)$. So assume $A \neq B$; we will prove $\wp(A) \neq \wp(B)$.

Since $A \neq B$, there exists an element x where either $x \in A$ and $x \notin B$ or vice-versa. Assume, without loss of generality, that $x \in A$ and $x \notin B$. Since $x \in A$, we see that $\{x\} \subseteq A$. Similarly, since $x \notin B$, we know $\{x\} \not\subseteq B$. This means that $\{x\} \in \wp(A)$ and $\{x\} \notin \wp(B)$, so $\wp(A) \neq \wp(B)$, as required. ■

Theorem: Let $f : A \to B$ be an injection. Then for any $X \subseteq A$, we have $X = f^{-1}[f[X]]$.

Theorem: Let $f : A \to B$ be an injection. Then for any $X \subseteq A$, we have $X = f^{-1}[f[X]]$.

Proof: Choose an $X \subseteq A$. We need to show that $X \subseteq f^{-1}[f[X]]$ and that $f^{-1}[f[X]] \subseteq X$. We already know the first of these is true, so we just need to show that the second is as well. So pick some $a \in f^{-1}[f[X]]$. We will prove that $a \in X$.

Since $a \in f^{-1}[f[X]]$, we know that $a \in A$ and $f(a) \in f[X]$. Because $f(a) \in f[X]$, we know there is some $a' \in X$ where $f(a') = f(a)$. We are assuming f is injective, so from $f(a') = f(a)$ we learn $a' = a$.

Putting everything together, we have that $a = a'$ and $a' \in X$, so $a \in X$, as needed. ■

Theorem: Let $f : A \to B$ be an injection. Then for any $Y \subseteq B$, we have $Y = f[f^{-1}[Y]]$.

Theorem: Let $f : A \to B$ be surjective. Then for any $Y \subseteq B$, we have $f[f^{-1}[Y]] = Y$.

Proof: Let $Y \subseteq B$ be chosen arbitrarily. We need to show that $f[f^{-1}[Y]] \subseteq Y$ and that $Y \subseteq f[f^{-1}[Y]]$. We already proved the first of these, so we just need to prove the second. So choose an arbitrary $b \in Y$; we will prove that $b \in f[f^{-1}[Y]]$.

Since $b \in Y$ and $Y \subseteq B$, we know that $b \in B$. Therefore, because f is surjective, there is some $a \in A$ where $f(a) = b$.

Since $f(a) = b$ and $b \in Y$, we see that $a \in f^{-1}[Y]$. Finally, because $a \in f^{-1}[Y]$ and $f(a) = b$, we see that $b \in f[f^{-1}[Y]]$, as required. ■